Math 261 B Tues. Oct. 20

Summary on highest weight modules

$T \subset B \subset G$

$B = T \ltimes U$

- Fin. dim'l G module ($= \mathcal{O}(G)$ comodule) $V \neq 0$ always contains a U invariant weight vector v_λ —

in particular, any vector of <u>maximal</u> weight

w.r.t. $\lambda < \mu \iff \mu - \lambda \in Q_+$

- Such a weight is <u>dominant</u> : $\langle \alpha_i^\vee, \lambda \rangle \geq 0 \quad \forall i$

- For each dominant $\lambda \longrightarrow$ standard module V^λ

$$= H^0(G/B, \mathcal{L}_{w_0(\lambda)})$$

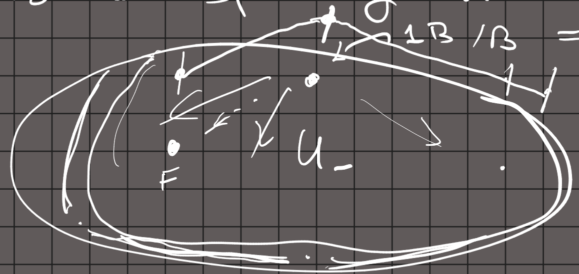
with maximal weight λ, $\dim (V^\lambda)_\lambda = 1 \qquad (V^\lambda)_\lambda = (V^\lambda)^U$

$w_0 \in W$ is the <u>longest element</u> : $\left[\, \ell(w) = |w(R_+) \cap R_-| \,\right]$

unique s.t. $w_0(R_+) = R_-$

λ dominant $\Rightarrow$ $w_0(\lambda)$ is antidominant $\langle \alpha_i^\vee, w_0(\lambda)\rangle \le 0$

Then $\mathcal{L}_{w_0(\lambda)} = G \times_B K_{w_0(\lambda)}$ has $H^0(G/B, \mathcal{L}_{w_0(\lambda)}) \neq 0$ with a 1-dimensional space of U_- invariants, of weight $w_0(\lambda)$



$U_- E$ is open, dense in G/B

- The submodule generated by $(V^\lambda)_\lambda$ is irreducible — this classifies the irreducibles.
- If char $k = 0$: V^λ is irreducible, and we have complete reducibility.

$$W = \bigoplus V^{\lambda_i}$$
$$W^U = \bigoplus \underbrace{(V^{\lambda_i})^U}_{\cong K_{\lambda_i}}$$

2 ways to prove complete reducibility:

- Construct Reynolds op. for $G(\mathbb{C})$ by integrating over a compact real form.
- Find <u>Casimir element</u> in $Z(\mathcal{U}(\mathfrak{g}))$, compute action on V^λ, discover that it's a different scalar for $\lambda \neq 0$ than for $\lambda = 0$.

How to get $G/B \hookrightarrow \mathbb{P}^N$

V^λ $\quad v_\lambda$ $\quad Kv_\lambda$ is a B-submodule

$$Kv_\lambda \,/\!/\, p = \mathbb{P}(V^\lambda) \text{ is a } B \text{ fixed point.}$$

$G/B \to \mathbb{P}(V^\lambda)$ gives a map.

$gB \mapsto gp$

If λ is dominant and <u>regular</u>: $\langle \alpha_i^\vee, \lambda \rangle > 0$ for all i

then $G/B \hookrightarrow \mathbb{P}(V^\lambda)$ $\quad$ trivial stabilizer in W.

In general, factors $G/B \to G/P \hookrightarrow \mathbb{P}(V^\lambda)$

$W \supset \mathrm{Stab}_W(\lambda) \quad \longleftrightarrow \quad P \supseteq B$

$G/P \hookrightarrow \mathbb{P}(V^\lambda)$ is a closed embedding.

<u>Examples</u> 1 SL_2 acts on $k[x,y]$ by linear changes of variable: $(x\ y) \mapsto (x\ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ $\quad x \mapsto tx$, $y \mapsto t^{-1}y$ $\quad x$ has weight 1, y has weight -1 $\quad x^k y^l$ has weight $k-l$

$\mathbb{Z} \ni m \quad t^m$

$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ $\quad x \mapsto x$, $y \mapsto y+bx$ $\quad f(x,y)$ U-invariant $\iff$ $f(x, y+bx) = f(x,y)$

U

$\iff f$ is a function of x.

$k[x,y]_d = V^d$ $\quad U$ invariants $= cx^d$, weight $= d$

$G/B = \mathbb{P}^1 = \mathbb{P}(k^2)$ $\quad \mathcal{L}_{-d} = \mathcal{O}(d)$ $\quad k[x,y]_d = H^0(\mathbb{P}^1, \mathcal{O}(d)) = H^0(G/B, \mathcal{L}_{w_0(d)})$

char $k=0$: V^d are irreducible, and their matrix entries form a basis of $\mathcal{O}(SL_2)$:

V^0

- $\bullet$

(1)

V^1

- x
- y

$x \mapsto ax + cy$

$y \mapsto bx + dy$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

V^2

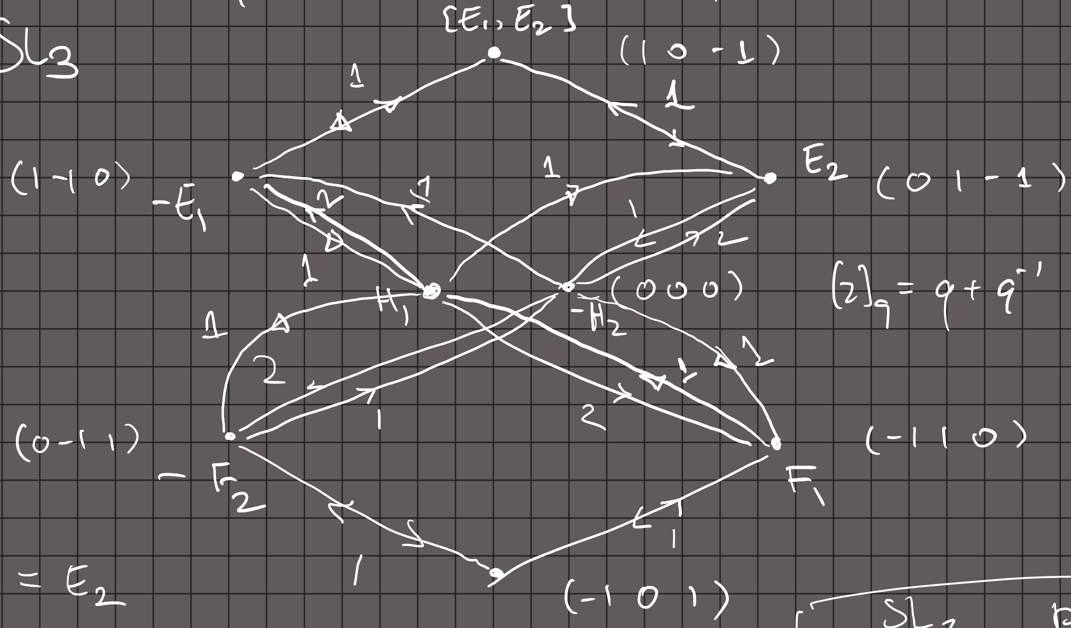
- $x^2 \mapsto a^2x^2 + 2ac\,xy + c^2y^2$
- $xy \mapsto ab\,x^2 + (ad+bc)\,xy + cd\,y^2$
- $y^2 \mapsto b^2x^2 + 2bd\,xy + d^2y^2$

$$\begin{pmatrix} a^2 & 2ac & c^2 \\ ab & ad+bc & cd \\ b^2 & 2bd & d^2 \end{pmatrix}$$

semisimple + indecomposable Cartan matrix

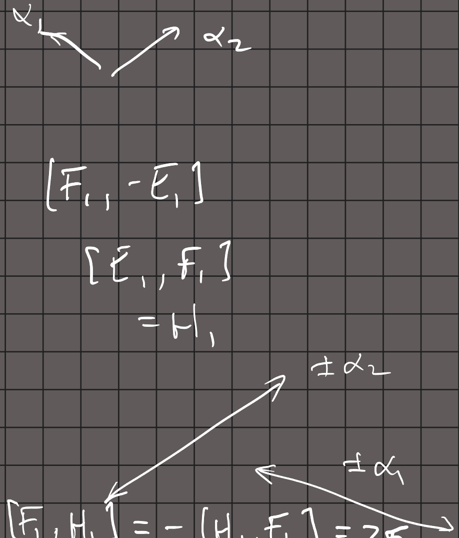
② If G is "simple", then $\mathfrak{g} = V^\lambda$ for λ the highest root (= unique dominant root).

Ex. sl_3



$[F_2, F_1]$

$[2]_q = q + q^{-1}$



$[F_1, -E_1]$

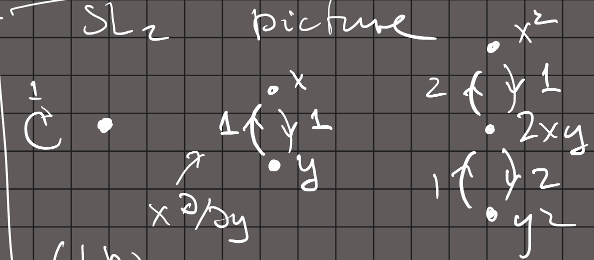
$[E_1, F_1] = H_1$

$[F_1, H_1] = -[H_1, F_1] = 2F_1$

$[E_2, H_1] = -[H_1, E_2] = E_2$

$$E_1 = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & \end{pmatrix} \qquad E_2 = \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

SL_2 picture



$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ is $y \mapsto y + bx = \exp(x\,\partial/\partial y)$

(3) $G = GL_n$ (or SL_n, PGL_n, ...)

$$G/B = \{ 0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset K^n \mid \dim(F_d) = d \}$$

Standard flag $E_\bullet$. $E_d = \langle e_1, \dots, e_d \rangle$

has stabilizer B.

Tautological line bundles

$\mathcal{L} = \Lambda^d F_d$ has fiber over E $\Lambda^d E_d$, spanned by $e_1 \wedge \cdots \wedge e_d$

$\shortparallel$

$\mathcal{L}_\lambda$ weight $\lambda = (1, \dots, 1, 0, \dots 0) = (1^d, 0, \dots, 0)$

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \qquad e_1 \wedge \cdots \wedge e_d \mapsto t_1 \cdots t_d$$

$\mathcal{L} = \Lambda^d (K^n / F_{n-d})$ over E $\Lambda^d (K^n / E_{n-d})$ spanned by $e_{n-d+1} \wedge \cdots \wedge e_n$

$\shortparallel$

$\mathcal{L}_\lambda$ $\lambda = (0, \dots, 0, 1^d)$

$$H^0(G/B, \mathcal{L}_{(0 \dots 0, 1^d)}) = H^0(G/B, \Lambda^d (K^n / F_{n-d})) = V_{(1^d, 0, \dots, 0)}$$

$W = S_n \qquad w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$

$V_{(1^d, 0, \dots, 0)} = \Lambda^d K^n$

defining rep

$e_{i_1} \wedge \cdots \wedge e_{i_d}$

weight $\quad e_1 \quad (0 \; 1 \; \cdots \;)$

$(1^d, 0 \dots 0)$ is the only dominant weight

$V_{(d, 0, \dots, 0)} = S^d K^n$